Stationary and nonstationary properties of evolving networks with preferential linkage

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Networks evolving by preferential attachment of both external and internal links are investigated. The rate of adding an external link is assumed to depend linearly on the degree of a preexisting node to which a new node is connected. The process of creating an internal link, between a pair of existing vertices, is assumed to be controlled entirely by the vertex that has more links than the other vertex in the pair, and the rate of creation of such a link is assumed to be, in general, nonlinear in the degree of the more strongly connected vertex. It is shown that degree distributions of networks evolving only by creating internal links display for large degrees a nonstationary power-law decay with a time-dependent scaling exponent. Nonstationary power-law behaviors are numerically shown to persist even when the number of nodes is not fixed and both external and internal connections are introduced, provided that the rate of preferential attachment of internal connections is nonlinear. It is argued that nonstationary effects are not unlikely in real networks, although these effects may not be apparent, especially in networks with a slowly varying mean degree.

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Structural and topological properties of many real complex systems emerging in various disciplines can be described by making use of networks which are neither static nor completely random [1–11]. These networks can easily be modeled by introducing to their creation processes different local mechanisms, such as the system evolution by adding new nodes and connections [12–14], preferential attachment of new vertices to existing vertices of large connectivities [1,4,5,15,16], the competition of vertices for links [17], aging of nodes [18,19], cost and size constraints [3,20], random removal of nodes [21,22], etc. This not only enables one to reproduce dynamic structures of various real networks, but also allows to indicate which mechanisms are responsible for macroscopic properties of particular network systems, and which mechanisms are important for controlling the formation of structures of these systems. It has recently been shown that a wide class of growing networks, including the World Wide Web (WWW) [23], the Internet [24], social networks [25], metabolic networks [10], and protein-protein interaction networks [11], reveal an ability to form self-organized structures, manifesting themselves in a scale-free power-law decay of the degree distribution. As has been proved, such a behavior of the degree distribution can be recovered by resorting to simple evolving models, in which new nodes are attached to randomly chosen existing nodes of high number of links, with the preference rate being a linear function of the degrees of preexisting nodes [5]. Although the growth of scale-free networks can also involve other local mechanisms [4], the network evolution by attaching new vertices, with the preference rate being linear in node degrees, appears to play the crucial role. However, current investigations of some real, evolving networks, especially the WWW [26], the Internet [19], and networks of scientific collaborations [8], show that the average degree for these networks increases with time. This suggests that, in addition to attaching new nodes and external links, such systems evolve by also creating internal links (between preexisting nodes). Indeed, the degree distribution of networks, which develops by adding external links with linear preferences as well as by introducing internal connections with bilinear preferences for pairs of high-degree nodes, has been shown analytically to display two different scaling regimes (short and long time) and, in the asymptotically long-time limit, has been proved to be entirely determined by internal links [8]. Moreover, it has been argued that the two time regimes of the power-law form of the degree distribution sustain even when the preference for attaching incoming nodes becomes nonlinear [8]. This is rather surprising since the nonlinearity in the preferential attachment of new nodes to networks evolving only by the creation of external links destroys the power-law form of the degree distribution [27–29].

Generally, the bivariate degree dependence of the rate of preferential creation of internal links is justified in modeling those real networks in which the attachment of a connection between two old nodes is determined by the activity and/or by the attractiveness of both the nodes. The most prominent examples of such webs are networks of scientific collaborations [8,25,30,31]. However, in many real networks, e.g., in the WWW, the introduction of internal links depends on the attractiveness (or the popularity) of single vertices rather than pairs of vertices, and then the appearance of a link between two existing vertices is governed by the more connected vertex in the pair.

Here, the effect of a preferential attachment of internal links, with the rate being dependent on degrees of single nodes, on the degree distribution is examined by means of numerical simulations of two evolving model networks. In the first network the number of nodes is fixed and the network evolves by adding internal links, while the second network is allowed to evolve by creating internal and external links as well. The rate of introducing an internal link between nodes i and j of degrees k_i and k_j is assumed to be

$$\Pi(k_i,k_j)=[\Theta(k_i-k_j)(k_i^{\nu}-k_j^{\nu})+k_j^{\nu}]/k_c^{\nu}, \qquad (1)$$

where $\Theta(x)=1$ if $x\geqslant 0$ and $\Theta(x)=0$ otherwise, $\nu>0$, and k_c denotes the maximal node degree at a current evolution

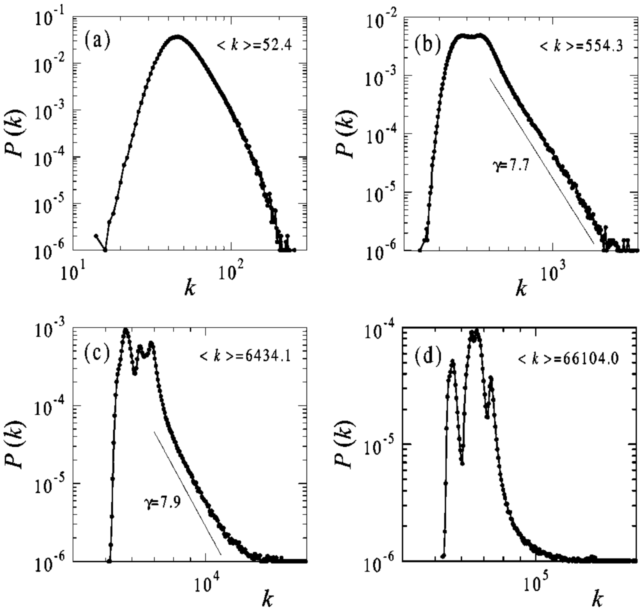

FIG. 1. Degree distributions for a fixed-node network of $N=10^5$ nodes, with the linear ($\nu=1$), single-node preferential rate of attaching links, at different evolution stages: (a) $m=10^2$. There is no power-law behavior. (b) $m=10^3$. A power-law behavior occurs for a large-degree tail. (c) $m=10^4$. A truncated power-law decay appears for large degrees. (d) $m=10^5$. The power-law decay does not emerge. The data were averaged over ten network realizations and were logarithmically binned.

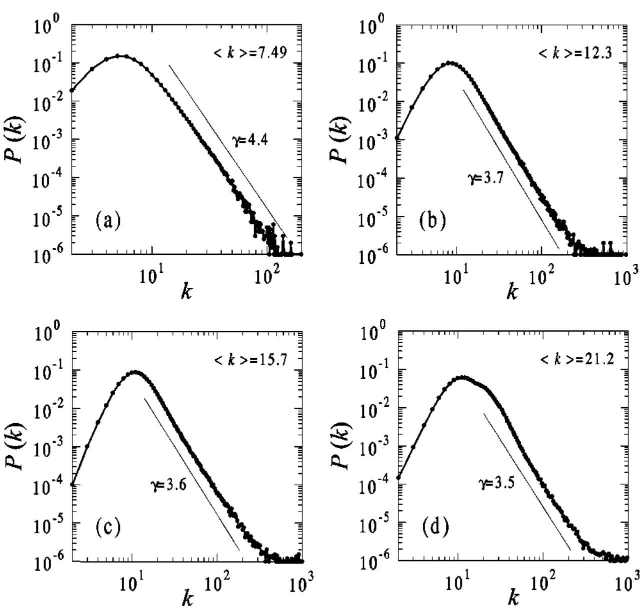

FIG. 2. Degree distributions in a fixed-node network of $N=10^5$ nodes, for a nonlinear ($\nu=1.5$), single-node preference rate, and for different evolution stages at which the power-law decay appears: (a) $m=10^2$, (b) $m=10^3$, (c) $m=10^4$, (d) $m=10^5$. The data were averaged over ten network constructions and were logarithmically binned.

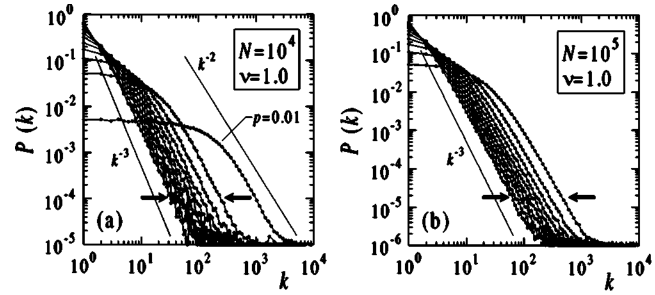

FIG. 3. Degree distributions for expanding networks with the linear ($\nu=1$), single-node attachment of internal links: (a) $N=10^4$, (b) $N=10^5$. The scaling exponent describing the power-law decay of the distributions for high degrees changes from $\gamma=3$ to $\gamma=2$ as p decreases from $p=1$ to $p=0$. The arrows indicate, from left to right, functions obtained for $p=1.0,0.9,\ldots,0.1$. The data, after averaging over ten network realizations, were logarithmically binned.

stage [32]. In both cases of models, the exponent ν is allowed to take values from a range between 0 and 2.

Fixed-node model. Starting with a homogeneous, connected network of N vertices, each of which is linked to two other vertices, at every time step, a randomly chosen pair of existing nodes is linked with the preference rate given by Eq. (1). The process of preferential attaching internal links is repeated m times. Consequently, the average degree $\langle k\rangle=1/N\Sigma_{i=1}^{N}k_i$ grows with time, while the structure of the network changes from homogeneous to inhomogeneous. Numerical results obtained for $0\leqslant\nu<1$ indicate that this model does not reveal any power-law behavior. However, when $\nu\geqslant1$, the degree distribution determined for the model exhibits a nonstationary power-law behavior $P(k)\sim k^{-\gamma}$ with a time-dependent scaling exponent γ, for a large-degree tail or for a truncated large-degree tail, when the mean degree exceeds a lower limit value and when it is less than an upper limit (Figs. 1 and 2). In the linear case ($\nu=1$), the power-law behavior emerges at a relatively large value of $\langle k\rangle$, in contrast to transitory scaling properties occurring at early stages of the evolution of fixed-node networks with a single-node linear preferential rate of attaching connections, but without links at the start [4]. In the nonlinear case, when $\nu>1$, the considered network model displays nonstationary power-law properties at small values of $\langle k\rangle$, with the scaling

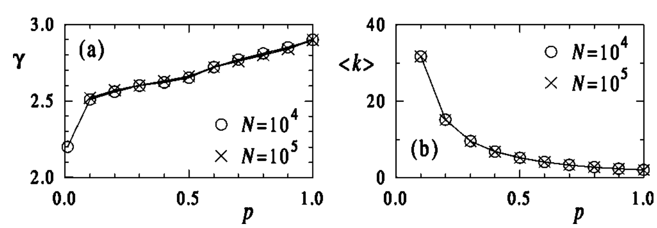

FIG. 4. Comparison of dependences of γ (a) and $\langle k\rangle$ (b) on p for different sizes of expanding networks in the case of the linear, single-node preferential attachment of internal links. The convergence of the respective dependences as N grows displays a scale-free (stationary) character of the power-law decay of the corresponding degree distribution.

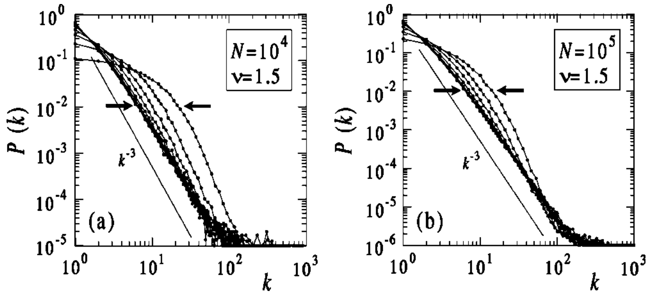

FIG. 5. Degree distributions for expanding networks with a nonlinear ($\nu=1.5$), single-node attachment of internal links: (a) $N=10^4$ and (b) $N=10^5$. The arrows indicate (from left to right) curves determined for $p=1.0,0.9,\ldots,0.1$ in the case of $N=10^4$, and for $p=1.0,0.9,\ldots,0.2$ in the case of $N=10^5$. Data were averaged over ten network realizations and were logarithmically binned.

index γ being smaller as compared to the linear case. As illustrated in Fig. 2, at least one of the quantities γ and $\langle k\rangle$ takes values inconsistent with the respective values found for real networks for $\nu>1$ [2]. In particular, both γ and $\langle k\rangle$ obtained for the fixed-node model with $\nu>1$ and for the WWW disagree [26,33].

Expanding model. Initially, the network consists of two connected vertices. At every time step, an external link is added with the probability $0<p\leqslant 1$, by attaching a new vertex to a randomly chosen vertex *i*, with a linear probability $\Pi(k_i)=k_i/k_c$, or an internal link is created with the probability $1-p$ by connecting a randomly chosen pair of old vertices, with a nonlinear (in general), single-node preference rate (1). Thus, in addition to preference rates, the creation of external and internal links depends on the probability *p*, in a manner that a large chance of introducing an external connection corresponds to an appropriately small chance of introducing an internal link, and conversely.

The degree distribution for the linear case ($\nu=1$) is shown in Fig. 3 for networks of different sizes and for different values of *p*. It follows that, for each $0<p\leqslant 1$, this distribution displays a power-law form (for sufficiently large degrees). Furthermore, both γ and $\langle k\rangle$ prove to be convergent for each *p* as the number of nodes grows. This is illustrated in Fig. 4 for two networks of very different sizes. Accordingly, for $\nu=1$, the power-law behavior of the model is scale-free (stationary). As Fig. 3(a) indicates, the scaling exponent crosses over for fixed *N* from the value $\gamma=3$ to $\gamma=2$ when $\langle k\rangle$ grows (or when $p\to 0$), similarly as in the case of time evolution of networks with bilinear preferential attachments of internal links [8]. When *p* decreases, the mean degree increases while the degree range, in which the power-law tail occurs, decreases and the deviation from the exact power-law behavior for small degrees becomes more pronounced. Finally, in the limit $p\to 0$, the expanding model converts into a fixed-node one and then its power-law behavior is no longer stationary. In the case when the rate of attaching internal links is nonlinear with $0<\nu<1$, the expanding model does not reveal a power-law behavior, just as the fixed-node model. However, if $\nu>1$, the degree distribution turns out to exhibit for each $p\in(0,1)$ a nonstationary power-law decay (Fig. 5), characterized by a time-dependent index γ [Fig. 6(a)], and associated with the mean degree, being constant or nearly constant in time [Fig. 6(b)]. When ν is large enough and *p* is sufficiently small, such a nonstationary power-law behavior persists for long time intervals, with $\gamma\simeq 3$ and $\langle k\rangle\in\langle 1,10\rangle$ (this is shown in Fig. 6 for $\nu=1.5$), in agreement with values of γ and $\langle k\rangle$ determined for real networks [2,26,34].

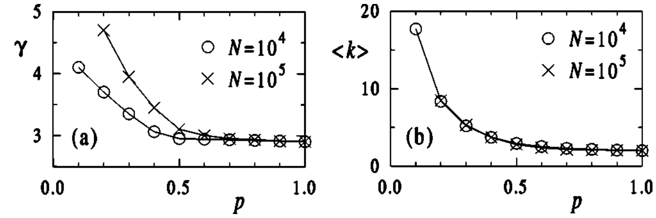

FIG. 6. Comparison of dependences of γ (a) and $\langle k\rangle$ (b) on *p* for expanding networks of different sizes in a nonlinear case $\nu=1.5$. The curves illustrate, for particular values of *p*, dependences of γ and $\langle k\rangle$ on *N*. In the case of $\langle k\rangle$, the dependence on the network size is very weak.

The preferential attachment of internal links in the considered network models is entirely governed by more connected nodes in pairs of joined, preexisting nodes. Certainly, the process of ‘‘attracting’’ not only new nodes but also old ones by highly connected (popular) nodes takes place in various real networks (e.g., in the WWW). For $0\leqslant\nu<1$, the creation of internal links reduces the inhomogeneity of the link distribution in expanding networks and, similarly as in the fixed-node networks, resists the emergence of the power-law decay of the degree distribution, even for the range of large degrees. If $\nu\geqslant 1$, the network inhomogeneity due to attaching internal links is sufficiently strong, in order to establish power-law behavior. However, the resulting power-law behavior is nonstationary both in fixed-node and expanding networks, except for expanding networks with $\nu=1$. As numerical results have shown, the time variation of the exponent γ determined for expanding networks with $\nu>1$ is very slow when the creation of internal links is much less probable than the attachment of external links. This indicates that nonstationary effects in real networks can be difficult to observe in relatively short time intervals and that, in the case of real networks with slowly varying mean degree (e.g., in the case of the WWW), the power-law behavior, which is believed to be stationary [2,4,5,14,23] can in fact be nonstationary.

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